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A Dynamical Calculation of the $\eta \to 3\pi$ Decay Rate

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We formulate dispersion relations for calculating the strength of the virtual $\eta \to \pi^0$ transition and include contributions from all possible intermediate baryon-antibaryon states. In our approximation, the change in isospin is derived from the baryon electromagnetic mass splittings, other contributory causes being neglected. The $\eta \to 3\pi$ decay rate is governed by the strength of this transition if, as is believed, the decay proceeds via virtual single pseudoscalar boson intermediate states. We recall that the contributions from the two sequences $\eta \to (\pi^0) \to \pi^+ + \pi^- + \pi^0$ and $\eta \to \pi^+ + \pi^- + (\eta) \to \pi^+ + \pi^- + \pi^0$ cancel each other exactly if the eightfold way interaction $4\pi\lambda(\pi\cdot\pi+\eta\eta)^2$ is used, and if the strength of the $\eta - \pi^0$ transition is taken to be independent of which particle is on its mass shell. Here this difficulty is resolved since our dynamical calculation allows for the mass dependence of the $\eta - \pi^0$ "black box." Comparison of our provisionally evaluated result with that of a previous calculation on $\eta \to 2\gamma$ gives an estimate for the $\eta \to 2\gamma / \eta \to \pi^+\pi^-\pi^0$ branching ratio which is in satisfactory agreement with experiment.

I. INTRODUCTION

THE decay of the η meson forms part of the general problem of electromagnetic effects in boson systems, and can thus be approached in two distinct ways. First, one can correlate the rates of some of the reactions by appeal to particular symmetry schemes (for the strong interactions) which accommodate the electromagnetic interactions in some definite and simple manner; this approach dispenses with assumptions about the specific mechanism for the individual reactions, but does depend entirely on the validity of the symmetry invoked, which in this paper is the eightfold way[1,2] realization of unitary symmetry. Second, one can attempt to make a dynamical, *ipso facto* model-dependent calculation for any particular reaction, which should, in principle, be independent of the validity of any symmetry scheme; here we shall assume that the η decays by primary virtual dissociation into bayon-antibaryon pairs.

Our hypothesis that baryon loops dominate defines a model which must ultimately be judged by experiment; to this end, calculations, on the same basis, of other radiative effects in pion systems are in progress. However, for $\eta \to 3\pi$ and some other decays there exist also tentative *a priori* arguments in favor of this mechanism; and we digress on them briefly, partly because the assumed dominance of such heavy intermediate states may appear strange at first sight. Therefore, we recall that as a rule exclusive concentration on the lightest intermediate state is appropriate only when seeking the *structure* of an amplitude, its absolute normalization being taken from experiment and incorporated as a subtraction constant. Here by contrast we try to calculate the subtraction constant itself; it is even conceivable that for this purpose the important states are those which dominate the spectral function at very high, rather than at very low, masses. Precisely this is illustrated in detail by the present model; one of our critical equations derives from the asymptotic behavior of the spectral function.

Such calculations[3] are all closely similar to the prototype Goldberger and Treiman approach[4] to charged pion decay. The dominant intermediate state is chosen by the light of the Lagrangian analogy, which favors those states whose constituents are believed to appear directly in the relevant weak coupling; in other words, supposedly large "numerators" take precedence over small "denominators" in the dispersion integrals. For the particular case of radiative effects in pion systems, it has been shown[5] that in a Lagrangian theory, with pions but no baryons, and with renormalizable electromagnetic couplings, an exact conservation law, that of amplitude parity,[6] forbids processes in which the total number of (real or virtual) pions changes from even to odd or odd to even. Amplitude parity conservation is broken by the pion-baryon couplings; thus A-forbiddenness does not reflect on the strength of a transition, but it does imply that an A-forbidden transition rate cannot be found from first principles without introducing baryons into the calculation at some stage. For instance, suppose we assumed that $\eta \to 3\pi$ was dominated by $\eta \to (\gamma+\rho) \to (\pi) \to 3\pi$, the last being a strong step. To complete the calculation in terms of fundamental constants we should need the electromagnetic $\rho \to \eta+\gamma$ and $\rho \to \pi+\gamma$ rates; thus, though correlations between the different pion-radiative ampli-

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[2] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).
[3] B. Barrett and G. Barton, Nuovo Cimento **29**, 703 (1963).
[4] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958); *ibid.* **111**, 354 (1958).
[5] G. Barton, Nuovo Cimento **27**, 1179 (1963).
[6] L. I. Schiff, *Proceedings of the International Conference on High Energy Physics at CERN, 1962* (CERN, Geneva, 1962), p. 690; and Phys. Rev. **130**, 458 (1963).

tudes may be made in this way, the chain can be anchored to the fundamental constants of the theory only when baryon states are introduced. The present calculation of $\eta \to 3\pi$ takes the simple-minded view, that failing contrary indications the baryons might as well be introduced immediately.[7]

The 2γ decay modes of the η and π^0 provide a clear example of processes which can be treated by the two methods outlined in the first paragraph, and to illustrate our general approach we recall here the results of previous work[3,8] on $\eta \to 2\gamma$. Cabibbo and Gatto[9] derived the relation

$$\mathfrak{M}(\eta \to 2\gamma) = (1/\sqrt{3})\mathfrak{M}(\pi^0 \to 2\gamma) \quad (1)$$

by considering the transformation properties of the electromagnetic current in the eightfold way; the consequent relation between the widths is

$$\Gamma(\eta \to 2\gamma) = \tfrac{1}{3}(m_\eta/m_{\pi^0})^3 \Gamma(\pi^0 \to 2\gamma). \quad (2)$$

Inserting the results for $\Gamma(\pi^0 \to 2\gamma)$ deduced from the most accurately observed values[10] of the π^0 lifetime, we obtain

$$\Gamma(\eta \to 2\gamma) \sim 77_{-16}^{+25} \text{ eV}, \quad (3a)$$

or

$$\Gamma(\eta \to 2\gamma) \sim 140_{-20}^{+30} \text{ eV}. \quad (3b)$$

The relation (2) may eventually be tested if the $\eta \to 2\gamma$ rate is measured by means of the Primakoff effect[11] and the discrepancy between the presently observed values of the π^0 lifetime is resolved. The sensitivity of the relation to deviations from unitary symmetry cannot, however, be estimated without a specific dynamical calculation, which is, of course, needed in any case for predicting the absolute values of the decay rates. Goldberger and Treiman[12] assumed that the π^0 decays by primary virtual disintegration into a nucleon-antinucleon pair which annihilates to form two photons, and obtained the result $\Gamma(\pi^0 \to 2\gamma) \approx 10$ eV using the observed values of the pion-nucleon coupling constant and the anomalous magnetic moments of the nucleons. We have rederived[3,13] the Goldberger-Treiman formula as a convergence condition on the dispersion integral,

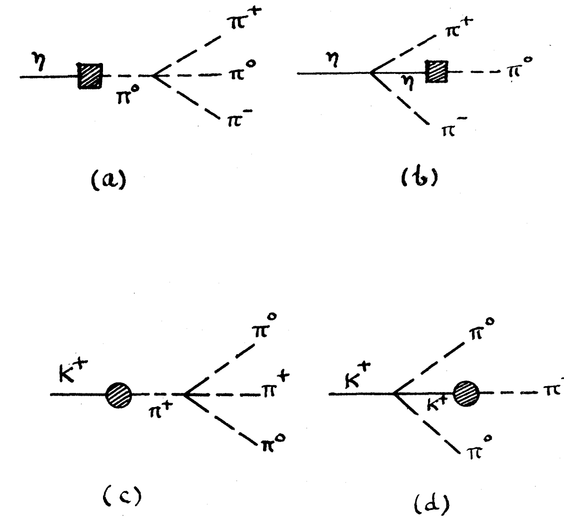

FIG. 1. Diagrams considered for $\eta \to \pi^+\pi^-\pi^0$ and $K^+ \to \pi^0\pi^0\pi^+$. In (a) and (b) the black box represents the electromagnetic $\eta-\pi^0$ transition, whereas in (c) and (d) the black circle represents the weak $K-\pi$ transition.

and extended it to include contributions from all possible baryon-antibaryon intermediate states; the analogous calculation for $\eta \to 2\gamma$ was also done. Since the pion-hyperon coupling constants, all the eta-baryon coupling constants, and the anomalous magnetic moments of the hyperons are not yet determined,[14] we cannot *in practice* evaluate our results without invoking some general symmetry which predicts these numbers. If we set all the baryon masses equal, use the eightfold way predictions[9,15] for the anomalous moments of the hyperons in terms of the neutron and proton moments, and use the eightfold way expressions[1] for the pion-baryon and eta-baryon couplings in terms of the parameter α (as in Sec. IV of this paper), our results for the $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$ matrix elements satisfy the relation (1) exactly. The insertion of the observed baryon masses does not affect this ratio very appreciably, and we find

$$\begin{aligned} \Gamma(\pi^0 \to 2\gamma) &\sim 2.5 \text{ to } 3.7 \text{ eV}, \\ \Gamma(\eta \to 2\gamma) &\sim 30 \text{ to } 60 \text{ eV} \end{aligned} \quad (4)$$

over the range of most reasonable values of α. We may expect, however, that deviations of the anomalous magnetic moments of the hyperons from the predicted values used may have a greater effect in violating the relation (1).

Before formulating and carrying out our dynamical calculation for the $\eta-\pi^0$ black box in Secs. III and IV,

[7] The A-parity considerations apply to $\eta \to 3\pi$ and $\pi^0 \to 2\gamma$, but not to $\eta \to 2\gamma$, since η has the quantum numbers of an assembly of four pions. Baryon dominance in $\eta \to 2\gamma$ represents a further extension of the hypothesis, which we have discussed elsewhere (Refs. 3, 13). Remark also that for $\pi^0 \to 2\gamma$ a model alternative to ours, namely, $\pi^0 \to (\omega+\rho) \to 2\gamma$, meets with difficulties when confronted with observation. [D. A. Geffen, Phys. Rev. **128**, 374 (1962), and references given there.]

[8] S. Okubo and B. Sakita, Phys. Rev. Letters **11**, 50 (1963).

[9] N. Cabibbo and R. Gatto, Nuovo Cimento **21**, 872 (1961).

[10] R. G. Glasser, N. Seeman, and B. Stiller, Phys. Rev. **123**, 1014 (1961), found $\tau_{\pi^0} = (1.9\pm0.5)\times10^{-16}$ sec; G. von Dardel, D. Dekkers, R. Mermod, J. D. van Putten, M. Vivargent, G. Weber, and K. Winter, Phys. Letters **4**, 51 (1963), found $\tau_{\pi^0} = (1.05\pm0.18)\times10^{-16}$ sec.

[11] C. M. Andersen, A. Halprin, and H. Primakoff, Phys. Rev. Letters **9**, 512 (1962); see also G. Bellettini, C. Bemporad, P. L. Braccini, L. Foà, and M. Toller, Phys. Letters **3**, 170 (1963).

[12] M. L. Goldberger and S. B. Treiman, Nuovo Cimento **9**, 451 (1958).

[13] B. Barrett and G. Barton, Phys. Letters **4**, 16 (1963).

[14] Two measurements have been made, both on the Λ, but they give conflicting results: R. L. Cook, E. W. Jenkins, T. F. Kycia, D. A. Hill, L. Marshall, and R. A. Schluter, Phys. Rev. **127**, 2223 (1962), reported $\mu_\Lambda = -1.5\pm0.5$ nucleon magnetons, whereas W. Kernan, T. B. Novey, S. D. Warshaw, and A. Wattenberg, Phys. Rev. **129**, 870 (1963), found $\mu_\Lambda = 0\pm0.6$ nm.

[15] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1961).

we consider in Sec. II the eightfold way predictions for the vertices involved in the over-all amplitude for the decay $\eta \to 3\pi$.

II. THE $\eta-\pi^0$ TRANSITION AND THE EIGHTFOLD WAY PREDICTIONS FOR $\eta \to 3\pi$

Kacser[16] has surveyed the arguments concerning the striking relationships observed[17] between the Dalitz plots for the three-pion decay modes of η, τ, and τ', and concluded that, barring coincidences, all these decays must proceed via a virtual intermediate single pion state.[18,19] The diagram of Fig. 1(a) has been considered by several authors[8,20]; a crude value for the $\pi^0 \to 3\pi$ vertex is obtained by using the interaction $4\pi\lambda(\pi\cdot\pi)^2$, where λ is the $\pi\pi$ coupling constant defined by Chew and Mandelstam.[21] [Ideally, λ should be replaced by the value which the analytic continuation of the $\pi\pi$ scattering amplitude assumes at the symmetric point $s_1=s_2=s_3=(3m_\pi^2+m_\eta^2)/3.$]

The $\eta-\pi^0$ black box involves two electromagnetic interactions (which change the isospin from 0 to 1), and contains diagrams similar to those which contribute to the electromagnetic self-masses of members of the octet of pseudoscalar mesons. Okubo and Sakita[8] have used the eightfold way relations[9]

$$\langle K^+|jj|K^+\rangle=\langle\pi^+|jj|\pi^+\rangle, \tag{6}$$

$$\langle K^0|jj|K^0\rangle=\langle\pi^0|jj|\pi^0\rangle-\sqrt{3}\langle\eta|jj|\pi^0\rangle, \tag{7}$$

to obtain

$$\begin{aligned}\gamma&=-\frac{1}{\sqrt{3}}[(m_{K^0}{}^2-m_{K^+}{}^2)+(m_{\pi^+}{}^2-m_{\pi^0}{}^2)]\\&=-[54\text{ MeV}]^2,\end{aligned} \tag{8}$$

where γ is the strength of the effective Lagrangian $\gamma\eta\pi^0$. Taking the value $\lambda=-0.18\pm0.05$ deduced by Hamilton *et al.*,[22] Okubo and Sakita give the result

$$\Gamma(\eta\to\pi^+\pi^-\pi^0)\approx142_{-70}{}^{+90}\text{ eV} \tag{9}$$

for the rate estimated by considering the contribution from Fig. 1(a) by itself. Hori *et al.*,[23] however, have pointed out that the diagram shown in Fig. 1(b), in which the single boson pole is an η, should also be included on an equal footing. Neither the structure of the $\eta\to3\pi$ Dalitz plot produced by the $\pi^0\to3\pi$ vertex in Fig. 1(a), nor the comparison with $K^\pm\to3\pi$ is necessarily affected by this observation[24] [since we must also include the additional diagram 1(d), as well as 1(c), for τ' decay]. Hori *et al.* note, however, that, with the eightfold way coupling $4\pi\lambda(\pi\cdot\pi+\eta\eta)^2$, the contributions from these two diagrams will cancel, since they differ only in the sign of the propagator. However, this conclusion is valid only if the strength of the $\eta-\pi^0$ transition is taken to be independent of the propagating momentum, i.e., of which particle is on its mass shell. We shall find that the cancellation seems to be adequately overcome, when, as in our dynamical calculation below, allowance is made for the mass dependence of the $\eta-\pi^0$ box.

III. THE LEHMANN REPRESENTATION FOR THE $\eta-\pi^0$ PROPAGATOR

In general it is difficult to formulate satisfactory dispersion relations when unstable particles are involved. For our particular problem the boson pole approximation[25] symbolized in Fig. 1(a) and (b) enables us to bypass these difficulties in the following way. The entire calculation proceeds, in principle, within the framework of renormalized perturbation theory, where adiabatic switching, at least of the electromagnetic couplings, is used to give meaning to the "incoming" single η state. (The η would be stable in absence of electromagnetic effects.) However, the strong four-boson vertices are taken to include all possible Feynman diagrams; hence their connection with the renormalized, i.e., observed, $\pi\pi$ amplitude or its analytic continuation. The $\eta-\pi^0$ black box is taken to include Feynman diagrams to all orders in all strong couplings, but only to leading, i.e., to first order in the fine structure constant, or alternatively, to first order in the observed electromagnetic baryon mass splittings. With these logical precautions the fact that the η is unstable does not intrude any further into the calculation of the $\eta-\pi^0$ black box, and from now on we can, and do, ignore it. This will be especially important in Eqs. (14) to (26) below.

Moreover, the perturbation theory framework identifies the black box unambiguously in relation to the Fourier transform of a time-ordered product of the

[16] C. Kacser, Phys. Rev. **130**, 355 (1963), and references given there.

[17] D. Berley, D. Colley, and J. Schultz, Phys. Rev. Letters **10**, 114 (1963).

[18] M. A. B. Bég and P. C. DeCelles, Phys. Rev. Letters **8**, 46 (1962).

[19] G. Barton and S. P. Rosen, Phys. Rev. Letters **8**, 414 (1962).

[20] Riazuddin and Fayyazuddin, Phys. Rev. **129**, 2337 (1963); **131**, 2840(E) (1963).

[21] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960). After earlier confusion, we do now believe that this is the correct normalization of the $\pi\pi$ interaction, and thank Professor Okubo for helpful correspondence.

[22] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

[23] S. Hori, S. Oneda, S. Chiba, and A. Wakasa, Phys. Letters **5**, 339 (1963). We are grateful to Professor Okubo for referring us to this paper before it was published.

[24] Hori *et al.* (Ref. 23) doubt whether the correspondence between the η, τ, and τ' Dalitz plots would continue to follow from boson-pole dominance in the exact eightfold way limit. We believe that this question cannot be plausibly settled at the present time, because it depends very delicately on cancellations between S and P wave components of the various four-boson vertices in the relevant diagrams, and because it will turn out below [see the comment following Eq. (70)] that even in the exact eightfold way limit these are not total cancellations. We feel that the problem merits more detailed consideration (it may even necessitate an estimate of the mass dependence of the $K-\pi$ black box), and may return to it in a subsequent publication.

[25] We make it plausible in the Appendix that the dominance of the boson-pole diagrams is compatible with our assumption of primary virtual dissociation into baryon-antibaryon loops.

interpolating fields. Let us denote it by a function $F(k^2)$ of the propagating momentum k; then, regarding for the moment both the η and the π^0 lines as internal lines, we have[26]

$$\frac{i}{k^2-m^2}F(k^2)\frac{i}{k^2-\mu^2}=G(k^2)\,, \tag{10}$$

where

$$G(k^2)=\int d^4(x-y)e^{-ik(x-y)}\langle 0|T(\boldsymbol{\eta}(x)\boldsymbol{\pi}^0(y))|0\rangle. \tag{11}$$

We express this Fourier transform in terms of a Lehmann[27] spectral function $\rho(a^2)$, and assume the unsubtracted form:

$$G(k^2)=\int da^2\rho(a^2)\frac{i}{k^2-a^2+i\epsilon}\,, \tag{12}$$

where

$$\rho(a^2)=(2\pi)^3\Sigma\langle 0|\eta|l,q\rangle\langle l,q|\pi^0|0\rangle\delta^4(q-a). \tag{13}$$

Here $\eta\equiv\boldsymbol{\eta}(0)$, and the sum is taken over the intermediate states l of four momentum q. We sum over all possible baryon-antibaryon ($B\bar{B}$) intermediate states as well as over the η and π^0 particles themselves, and have

$$\begin{aligned}\rho(a^2)=&(2\pi)^3\Sigma\langle 0|\eta|\eta,q_\eta\rangle\langle\eta,q_\eta|\pi^0|0\rangle\delta^4(q_\eta-a)\\&+(2\pi)^3\Sigma\langle 0|\eta|\pi^0,q_\pi\rangle\langle\pi^0,q_\pi|\pi^0|0\rangle\delta^4(q_\pi-a)\\&+\sigma(a^2)\,,\end{aligned} \tag{14}$$

where $\sigma(a^2)$ is just the contribution from the $B\bar{B}$ intermediate states; the lightest of these is the proton-antiproton pair of mass $2M_p$.

Carrying out the summation in (14), we find

$$\rho(a^2)=R_\eta\delta(a^2-m^2)+R_\pi\delta(a^2-\mu^2)+\sigma(a^2)\,, \tag{15}$$

where

$$R_\eta=\langle\eta,q|\pi^0|0\rangle(V2q_0)^{1/2}\,, \tag{16}$$

$$R_\pi=\langle 0|\eta|\pi^0,q\rangle(V2q_0)^{1/2} \tag{17}$$

(V is the normalization volume). Inserting (15) into (12), we have

$$G(k^2)=\frac{i}{k^2-m^2}R_\eta+\frac{i}{k^2-\mu^2}R_\pi+\int_{4M_p^2}^{\infty}da^2\sigma(a^2)\frac{i}{k^2-a^2+i\epsilon}. \tag{18}$$

With the eightfold way boson coupling $4\pi\lambda(\pi\cdot\pi+\eta\eta)^2$, the contribution of the diagrams of Fig. 1(a) and (b) to the invariant matrix element for $\eta\to\pi^+\pi^-\pi^0$ is just

$$32\pi\lambda\left\{\left[F(k^2)\frac{i}{k^2-\mu^2}\right]\bigg|_{k^2=m^2}+\left[F(k^2)\frac{i}{k^2-m^2}\right]\bigg|_{k^2=\mu^2}\right\}=32\pi\lambda[R_\eta+R_\pi]\equiv 32\pi\lambda R \tag{19}$$

[and we have exact cancellation if $F(\mu^2)=F(m^2)=\gamma$].

Our formalism, then, involves two unknowns, the strengths R_η and R_π for the $\eta-\pi^0$ transition when the η or the π^0 are, respectively, on their mass shells. Although, with the pure eightfold way boson coupling used above, we ultimately require only the sum $R=R_\eta+R_\pi$ obtained in Eq. (19), we shall find that R_η and R_π enter again, and separately, in our approximation for $\sigma(a^2)$, so that we do in fact need two simultaneous equations, which we can obtain by considering the canonical equal time commutation rules of the η and π^0 fields and their sources.

For the vacuum expectation value of the commutator of the η and π^0 fields we write

$$\langle 0|[\boldsymbol{\eta}(x),\boldsymbol{\pi}^0(y)]|0\rangle\equiv i\Delta'(x-y)=\int da^2\rho(a^2)i\Delta(x-y|a^2)\,, \tag{20}$$

where $\Delta(x-y|a^2)$ is the usual Δ function for free particles of mass a. Taking $y=0$, differentiating with respect to x_0, and setting $x_0=0$, we have

$$\left\langle 0\left|\left[\frac{\partial\boldsymbol{\eta}}{\partial x_0}(\mathbf{x},0),\boldsymbol{\pi}^0(0)\right]\right|0\right\rangle=\int da^2\rho(a^2)i\frac{\partial}{\partial x_0}\Delta(x|a^2)\bigg|_{x_0=0}=\int da^2\varphi(a^2)i\delta(\mathbf{x}). \tag{21}$$

Since this equal time commutator for two different fields vanishes,[28] we have found

$$0=\int da\,\rho(a^2). \tag{22}$$

The second of the two equations needed to calculate R_π and R_η could be obtained along parallel lines by considering the commutator of the η with the *source* J_π of the π^0 field: we use the equations

$$(\Box_y{}^2+\mu^2)\boldsymbol{\pi}^0(y)=J_\pi(y)\,, \tag{23}$$

$$(\Box_y{}^2+a^2)\Delta(x-y|a^2)=0\,, \tag{24}$$

[26] *Notation:* We use natural units $\hbar=c=1$, unrationalized coupling constants $e^2/4\pi=1/137$, $g^2/4\pi=15$ etc., and the metric $g^{00}=-g^{11}=-g^{22}=-g^{33}=1$. The unlabeled letters M, m, μ are reserved for the masses of the nucleon, eta, and neutral pion, respectively; the letters M and m with suffixes are used for the masses of other particles.

[27] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

[28] If we were to regard the η as a composite particle built up from the pion fields, we should argue that the equal time commutators in (21) and (26) are small, i.e., of the order of magnitude of the $\pi\pi$ coupling constant λ; then, since the matrix element $\mathfrak{M}(\eta\to 3\pi)\propto R\lambda$, our neglect of the equal time commutators is equivalent to carrying out the calculation of $\mathfrak{M}(\eta\to 3\pi)$ to first order in λ. Alternatively, if we regard all operators as ordered products (which is in any case necessary for self-consistency), then the relevant equal time vacuum expectation values all vanish.

to write

$$\langle 0|[\boldsymbol{\eta}(x),J_\pi(y)]|0\rangle = i\int da^2\rho(a^2)(-a^2+\mu^2)\Delta(x-y|a^2)\,, \quad (25)$$

and again differentiate with respect to x_0 to obtain

$$\left\langle 0\left|\left[\frac{\partial\boldsymbol{\eta}}{\partial x_0}(\mathbf{x},0),J_\pi(0)\right]\right|0\right\rangle, = i\int da^2\rho(a^2)(-a^2+\mu^2)\delta(\mathbf{x}) \quad (26)$$

giving[28]

$$0=\int da^2(a^2-\mu^2)\rho(a^2)\,. \quad (27)$$

From Eqs. (15), (22), and (27) we derive the two basic equations

$$R_\eta+R_\pi+\int da^2\sigma(a^2)=0\,, \quad (28)$$

$$m^2R_\eta+\mu^2R_\pi+\int da^2a^2\sigma(a^2)=0\,. \quad (29)$$

At this point it is convenient to digress somewhat in order to forestall the following difficulty that may be encountered if one proceeds directly from these equations. One of the crucial relations to be used below is the convergence condition on (29), i.e.,

$$\lim_{a^2\to\infty} a^4\sigma(a^2)=0\,. \quad (30)$$

But taken together with (28) and (29) itself this makes three equations for the two unknowns R_η and R_π (since the R's themselves occur in σ), which may be mutually inconsistent. Such an inconsistency can be avoided by a method originally devised to deal with a similar situation in the Goldberger-Treiman formulas for $\pi^+\to\mu\nu$ and $\pi^0\to 2\gamma$; we refer to our previous paper[3] for a detailed physical interpretation of the procedure in these closely analogous cases.

The method is best displayed by refocussing attention on the propagators. From this viewpoint (22) can be written as

$$\lim_{k^2\to\infty} k^2G(k^2)=\int da^2\rho(a^2)=0\,, \quad (31)$$

while (29) becomes

$$\lim_{k^2\to\infty} k^2\tilde{G}(k^2)=\int da^2a^2\rho(a^2)=0\,, \quad (32)$$

where $\tilde{G}(k^2)$ is the mixed propagator constructed from η and $\Box^2\pi^0$. Underlying (32) is the assumption that $\tilde{G}(k^2)$ satisfies an unsubtracted Lehmann representation:

$$\tilde{G}(k^2)=\int da^2a^2\rho(a^2)/(k^2-a^2)\,. \quad (33)$$

Moreover, (32) shows that $\tilde{G}(k^2)$ vanishes at infinity faster than $1/k^2$, i.e., faster than (33) would indicate explicitly at first sight. The crucial procedure in avoiding inconsistencies is to relax by one step the requirements on the behavior of $\tilde{G}(k^2)$ at infinity. Thus, instead of (32), we demand only that $\lim_{k^2\to\infty}[k^2\tilde{G}(k^2)]$ exist as a finite constant. In view of (33) this implies that $\int da^2a^2\rho(a^2)<\infty$, whence

$$\lim_{a^2\to\infty} a^4\rho(a^2)=0\,. \quad (34)$$

Equation (34) is the same as the convergence condition (30) of the original version. However, no inconsistency can now occur, since one of the three equations is taken up in determining the otherwise undefined numerical value of $\lim_{k^2\to\infty}[k^2\tilde{G}(k^2)]$, which is of no further relevance to the problem in hand. We should note also that the development leading to Eq. (22) is not affected by the question of the asymptotic behavior of $\tilde{G}(k^2)$.

At the start of this important digression on the problems of convergence and inconsistency, we referred to our earlier paper[3] in which similar problems were first encountered. It is worth emphasizing, however, that in contrast to the case of the Goldberger-Treiman formula for $\pi^0\to 2\gamma$, the convergence condition here does *not* uniquely determine the numbers of interest, R_η and R_π. Thus, even if the high-energy behavior of the spectral function $\sigma(a^2)$ is well approximated by our model, the results remain sensitive to the behavior of σ at low energy.

IV. EVALUATION OF THE SPECTRAL FUNCTION

For clarity, we first perform the calculation with nucleon-antinucleon states only, and then extend the result to include contributions from all baryon-antibaryon states.

We write the neutron-antineutron and proton-antiproton contributions explicitly

$$\sigma(a^2)=(2\pi)^3\sum[\langle 0|\eta|n\bar{n}\rangle\langle n\bar{n}|\pi^0|0\rangle\delta^4(a-n-\bar{n}) +\langle 0|\eta|p\bar{p}\rangle\langle p\bar{p}|\pi^0|0\rangle\delta^4(a-p-\bar{p})]\,. \quad (35)$$

Here $\sum$ implies summation over momenta and spins, and half the sum of $|\text{in}\rangle$ and $|\text{out}\rangle$ states is taken to ensure that the correct reality conditions are satisfied automatically in this approximation. If isospin were conserved the $n\bar{n}$ and $p\bar{p}$ contributions would cancel; apart from a contribution in terms of R_η and R_π themselves, we shall take into account only those i-violating effects which derive from the observed baryon electromagnetic mass splittings. The restriction to the mass-splitting contributions is inspired by convenience; it is made partly to illustrate and test the model and partly to provide a first estimate of the result. In order to see roughly what the approximation involves, consider the diagrams containing a closed baryon loop, and a photon line anchored at both ends to the baryon lines. Out of these diagrams we take into account those in which the photon line gives rise to a baryon self-energy insert; we leave out those diagrams in which the photon line leads to a "renormalization" of a (strong) vertex. A more dispersion-theoretic way of phrasing this is, that in our spectral function we allow for the phase space differences dictated by the observed neutron-proton mass splitting, but take no account of any other electromagnetic contributions to nucleon-antinucleon scattering, such as the Coulomb interaction between p and $\bar{p}$.

We define

$$\langle 0|J_\pi|n\bar{n}\ \text{in}\rangle =\left(\frac{M_n^2}{V^2n_0\bar{n}_0}\right)^{1/2}\bar{v}(\bar{n})\gamma_5(gK_n\tau^0+P_n1)u(n)\,, \quad (36)$$

$$\langle 0|J_\pi|p\bar{p}\ \text{in}\rangle =\left(\frac{M_p^2}{V^2p_0\bar{p}_0}\right)^{1/2}\bar{v}(\bar{p})\gamma_5(gK_p\tau^0+P_p1)u(p)\,, \quad (37)$$

where $J_\pi(x)=(\Box^2+\mu^2)\pi^0(x)$. Here we are considering the π to be a particle of definite isospin $I=1$, and are explicitly separating the $\pi^0n\bar{n}$ and $\pi^0p\bar{p}$ form factors into the strong, isospin-conserving parts $K_n[(n+\bar{n})^2]$ and $K_p[(p+\bar{p})^2]$, and into the electromagnetic, isospin-violating parts $P_n[(n+\bar{n})^2]$ and $P_p[(p+\bar{p})^2]$. We take $K_n(\mu^2)=K_p(\mu^2)=1$, so that g is just the pion-nucleon coupling constant (and $P_n(\mu^2)=P_p(\mu^2)$ is the correction δg considered by Riazuddin and Fayyazuddin[20]). Similarly, we define

$$\langle 0|J_\eta|n\bar{n}\ \text{in}\rangle =\left(\frac{M_n^2}{V^2n_0\bar{n}_0}\right)^{1/2}\bar{v}(\bar{n})\gamma_5(\tilde{g}L_n1+H_n\tau^0)u(n)\,, \quad (38)$$

$$\langle 0|J_\eta|p\bar{p}\ \text{in}\rangle =\left(\frac{M_p^2}{V^2p_0\bar{p}_0}\right)^{1/2}\bar{v}(\bar{p})\gamma_5(\tilde{g}L_p1+H_p\tau^0)u(p)\,, \quad (39)$$

where $J_\eta(x)=(\Box^2+m^2)\eta(x)$. Here the L's are the strong interaction form factors, normalized to $L_n(m^2)=L_p(m^2)=1$, and $\tilde{g}$ is the eta-nucleon coupling constant (analogous to g).

The summation over spins and momenta involved in Eq. (23) gives (putting $s=a^2$)

$$\sigma(s)=\sigma_n(s)+\sigma_p(s)\,, \quad (40)$$

where

$$\sigma_n(s)=\frac{1}{8\pi^2 s}\left(\frac{s-4M_n^2}{s}\right)^{1/2} \quad (41)$$
$$\times\text{Re}[(\tilde{g}L_n(s)-H_n(s))(P_n^*(s)-gK_n^*(s))]\,,$$

$$\sigma_p(s)=\frac{1}{8\pi^2 s}\left(\frac{s-4M_p^2}{s}\right)^{1/2} \quad (42)$$
$$\times\text{Re}[(\tilde{g}L_p(s)+H_p(s))(P_p^*(s)+gK_p^*(s))]\,.$$

We combine these equations to write σ in two parts; where the $n\bar{n}$ and $p\bar{p}$ contributions tend to cancel, we write

$$\sigma_1(s)=\frac{\tilde{g}g}{8\pi^2}\text{Re}[L(s)K^*(s)] \times\frac{1}{s}\left\{\left(\frac{s-4M_p^2}{s}\right)^{1/2}-\left(\frac{s-4M_n^2}{s}\right)^{1/2}\right\} \quad (43)$$

and neglect the distinction between L_n and L_p, K_n and K_p, i.e., we work to first order in isospin-violating terms. For the same reason, in the second part we neglect the neutron-proton mass difference, and write

$$\sigma_2(s)=\frac{1}{4\pi^2}\text{Re}[gH(s)K^*(s)+\tilde{g}L(s)P^*(s)] \times\frac{1}{s}\left(\frac{s-4M^2}{s}\right)^{1/2} \quad (44)$$

(M is the mean nucleon mass). We also neglect the remaining second-order terms $H(s)P^*(s)$.

The evaluation of the form factors is similar to the dispersion relations treatment of Goldberger and Treiman[12] for the $\pi^0\to2\gamma$ amplitude. We define

$$J_{\eta,p}=\left(\frac{V^2p_0\bar{p}_0}{M_p^2}\right)^{1/2}\langle 0|J_\eta|p\bar{p}\ \text{in}\rangle =\bar{v}(\bar{p})\gamma_5(\tilde{g}L+H)u(p)\,, \quad (45)$$

and in the standard way obtain the absorptive part

$$A_{J_{\eta,p}}=-\frac{1}{2}\left(\frac{Vp_0}{M_p}\right)^{1/2}\sum_{\mathbf{q}}\bar{v}(\bar{p})\langle 0|J_\eta|q\rangle \times\langle q|f|p\rangle(2\pi)^4\delta^4(q-p-\bar{p})\,. \quad (46)$$

In the sum over intermediate states we include the single pion, and proton-antiproton and neutron-antineutron pairs; then

$$A_{J_{\eta,p}}=-\frac{1}{2}\left(\frac{Vp_0}{M_p}\right)^{1/2}(2\pi)^4 \times\{\Sigma\langle 0|J_\eta|\pi^0,q\rangle\bar{v}(\bar{p})\langle\pi^0,q|f|p\rangle\delta^4(q-p-\bar{p}) \quad (47)$$
$$+\Sigma\langle 0|J_\eta|p'\bar{p}'\rangle\bar{v}(\bar{p})\langle p'\bar{p}'|f|p\rangle\delta^4(p'+\bar{p}'-p-\bar{p})$$
$$+\Sigma\langle 0|J_\eta|n'\bar{n}'\rangle\bar{v}(\bar{p})\langle n'\bar{n}'|f|p\rangle\delta^4(n'+\bar{n}'-p-\bar{p})\}\,.$$

In the first term we use the definition (16) and obtain the contribution

$$\mathrm{Im}[\tilde{g}L(s)+H(s)]^{\pi}=-\pi gR_{\pi}(m^2-s)\delta(s-\mu^2). \quad (48)$$

The second term in (47) involves $J_{\eta,p}$ itself, and we have the familiar result

$$\mathrm{Im}[\tilde{g}L(s)+H(s)]^{p\bar{p}}=\mathrm{Re}[(\tilde{g}L(s)+H(s))\beta_p^*], \quad (49)$$

where

$$\beta_p=e^{i\delta_p}\sin\delta_p\,, \quad (50)$$

and δ_p is the phase shift for proton-antiproton scattering in the ${}^1S_0$ state. The third term in (47) would similarly be expressed in terms of the phase shift for charge-exchange ($p\bar{p}\rightarrow n\bar{n}$) scattering; we note that this is observed to be very small[29] and therefore neglect the contribution to $A_{J_{\eta,p}}$ of the $n\bar{n}$ states.

We combine the contributions of (48,49) to write

$$\mathrm{Im}[\tilde{g}L+H]=-\pi gR_{\pi}(m^2-s)\delta(s-\mu^2) +\mathrm{Re}[(\tilde{g}L+H)\beta_p^*]; \quad (51)$$

the corresponding treatment of the matrix element $\langle 0|J_\eta|n\bar{n}\text{ in}\rangle$ gives

$$\mathrm{Im}[\tilde{g}L-H]=+\pi gR_{\pi}(m^2-s)\delta(s-\mu^2) +\mathrm{Re}[(\tilde{g}L-H)\beta_n^*], \quad (52)$$

where $\beta_n=e^{i\delta_n}\sin\delta_n$, and δ_n is the ${}^1S_0$ phase shift for $n\bar{n}$ scattering. Adding these equations (and neglecting second-order isospin-violating terms), we have

$$\mathrm{Im}L=\tfrac{1}{2}\,\mathrm{Re}[L(\beta_p^*+\beta_n^*)]=\mathrm{Re}[L\beta^*]; \quad (53)$$

this is the standard result of the approximation which ascribes the s dependence of the strong interaction form factor entirely to nucleon-antinucleon rescattering. The solution of the subtracted dispersion relation

$$L(s)=1+\frac{s-m^2}{\pi}\int ds'\frac{\mathrm{Im}L(s')}{(s'-m^2)(s'-s-i\epsilon)} \quad (54)$$

is just the Omnès[30] function

$$L(s)=\exp\left\{\frac{s-m^2}{\pi}\int ds'\frac{\phi(s')}{(s'-m^2)(s'-s-i\epsilon)}\right\} \quad (55)$$

(with $\tan\phi=\mathrm{Re}\beta/(1-\mathrm{Im}\beta)$.) We refer to the arguments given in the paper[3] on the Goldberger-Treiman formulas to justify the approximation $L(s)\approx 1$ which we shall again make here.

Subtracting (52) from (51), we obtain the equation for the first-order i-violating terms:

$$\mathrm{Im}H=-\pi gR_{\pi}(m^2-s)\delta(s-\mu^2)+\mathrm{Re}(HB^*) +\tfrac{1}{2}\mathrm{Re}[\tilde{g}L(\beta_p^*-\beta_n^*)]. \quad (56)$$

We postulate an unsubtracted dispersion relation for $H(s)$, which gives

$$H(s)=gR_{\pi}\frac{(m^2-\mu^2)}{(s-\mu^2)}+\frac{1}{\pi}\int ds'\frac{\tan\phi(s')\,\mathrm{Re}H(s')}{s'-s-i\epsilon} +\frac{\tilde{g}}{2\pi}\int ds'\frac{\tan\phi_p(s')-\tan\phi_n(s')}{s'-s-i\epsilon}. \quad (57)$$

We again argue[3] that the low angular momentum phase shifts for nucleon-antinucleon scattering should be predominantly imaginary, so that ϕ is small and we can approximate the Omnès exponentials by unity, i.e., that we can drop the second term in (57). We need, however, to compare the third term with the first (pion pole) term, and a very rough estimate[31] suggests that the third term can also be discarded. We thus have

$$H(s)=gR_{\pi}\frac{m^2-\mu^2}{s-\mu^2}\,, \quad (58)$$

and the analogous result

$$P(s)=\tilde{g}R_{\eta}\frac{\mu^2-m^2}{s-m^2}\,, \quad (59)$$

for the isospin-violating part P of the pion-nucleon form factor.

We insert the results (58,59) and the approximations $L(s)=K(s)=1$ into Eqs. (43,44) for the contributions to the spectral function, and obtain [dropping m^2, $\mu^2(\ll s\leqslant 4M^2)$ in the denominators]

$$\sigma(s)=\frac{1}{8\pi^2 s}\left[g\tilde{g}\left\{\left(\frac{s-4M_p^2}{s}\right)^{1/2}-\left(\frac{s-4M_n^2}{s}\right)^{1/2}\right\} +\frac{2(m^2-\mu^2)}{s}\left(\frac{s-4M^2}{s}\right)^{1/2}(g^2R_{\pi}-\tilde{g}^2R_{\eta})\right]. \quad (60)$$

To first order in the mass difference $\delta M=M_n-M_p$, the basic equations (22) and (34) (with $s=a^2$) give

$$0=R_{\eta}+R_{\pi}+\frac{\tilde{g}g}{4\pi^2}\frac{\delta M}{M}+\frac{m^2-\mu^2}{3M^2}(g^2R_{\pi}-\tilde{g}^2R_{\eta})\,, \quad (61)$$

and

$$0=4\tilde{g}gM\delta M+2(m^2-\mu^2)(g^2R_{\pi}-\tilde{g}^2R_{\eta})\,, \quad (62)$$

from which we obtain the solutions

$$R_{\eta}=\frac{\tilde{g}g}{\tilde{g}^2+g^2}\left[+\frac{2M\delta M}{m^2-\mu^2}-\frac{1}{6\pi^2}g^2\frac{\delta M}{M}\right], \quad (63)$$

$$R_{\pi}=\frac{g\tilde{g}}{g^2+\tilde{g}^2}\left[-\frac{2M\delta M}{m^2-\mu^2}-\frac{1}{6\pi^2}\tilde{g}^2\frac{\delta M}{M}\right], \quad (64)$$

which give

$$R=-\frac{1}{6\pi^2}g\tilde{g}\frac{\delta M}{M}. \quad (65)$$

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[29] J. G. Loken and M. Derrick, Phys. Letters 3, 334 (1963).
[30] R. Omnès, Nuovo Cimento 8, 326 (1958).
[31] B. Barrett, thesis, Oxford University, 1963 (unpublished).

We now add in the contributions to the spectral function from all the other $B\bar{B}$ states[32]; Eq. (60) is replaced by

$$\sigma(s)=\frac{1}{8\pi^2 s}\left[\sum_i g_{\eta i}g_{\pi i}\left\{\left(\frac{s-4M_{i+}^2}{s}\right)^{1/2}-\left(\frac{s-4M_{i-}^2}{s}\right)^{1/2}\right\}+\frac{2(m^2-\mu^2)}{s}\sum_j\left(\frac{s-4M_j^2}{s}\right)^{1/2}\xi_j(g_{\pi j}^2R_\pi-g_{\eta j}^2R_\eta)\right]. \tag{66}$$

(The factors ξ_j take count of the multiplicities of the different baryon multiplets.) In the first sum, which represents the "inhomogeneous" term of our equation, the violation of isospin is caused by the electromagnetic mass splittings of the baryon multiplets, and we have contributions from $N\bar{N}$, $\Sigma\bar{\Sigma}$ and $\Xi\bar{\Xi}$ states; we have written M_{i+}, M_{i-} for the masses of the components of larger, smaller z components of isospin. In the second sum, which represents a "homogeneous" or "damping" term, we have contributions from *all* $B\bar{B}$ states which can be coupled strongly to either the η or the π^0; the $\Lambda\bar{\Lambda}$ state, for example, which is coupled strongly to the η, is also coupled electromagnetically to the π^0 by the η pole term, and gives a term proportional to $g_{\eta\Lambda\Lambda}^2R_\eta$.

Inserting (66) into (22) and (34), we find

$$R_\eta=\frac{\sum_i g_{\eta i}g_{\pi i}\frac{\delta M_i}{M_i}\left[+\frac{2M_i^2}{m^2-\mu^2}-\frac{(\sum_j\xi_j g_{\pi j}^2)}{6\pi^2}\right]}{\sum_j\xi_j(g_{\pi j}^2+g_{\eta j}^2)} \tag{67}$$

and an analogous equation for R_π [compare (63) and (64)]; their sum is just

$$R=-\frac{1}{6\pi^2}\sum_i g_{\eta i}g_{\pi i}\frac{\delta M_i}{M_i}. \tag{68}$$

We note that this very simple result is not very different from that obtained by including only the mass difference contributions to $\sigma(s)$, i.e., dropping the damping terms, and using (22) alone, which would give

$$R=-\frac{1}{4\pi^2}\sum_i g_{\eta i}g_{\pi i}\frac{\delta M_i}{M_i}. \tag{69}$$

There is no trace of saturation in (68), i.e., there are no coupling constants in the denominators, contrary to the characteristic appearance of the Goldberger-Treiman formulas[4,12] for π^+ and π^0 decay. *Moreover, R is independent of the η and π^0 masses.* This is perhaps not so surprising if we recall Eq. (19); we have to calculate

$$\frac{F(m^2)}{m^2-\mu^2}+\frac{F(\mu^2)}{\mu^2-m^2}=\left.\frac{dF(x)}{dx}\right|_{x=\mu^2}+\text{ terms of higher order in }(m^2-\mu^2), \tag{70}$$

and the expansion may be justified since the characteristic mass involved in $\sigma(s)$ is $M^2\gg\mu^2$.

It is amusing to envisage the situation that would obtain in the hypothetical limit as unitary symmetry becomes exact. The denominators $(m^2-\mu^2)$ would then approach zero, but they would be cancelled exactly by the corresponding factor in the numerator $(dF/d\mu^2)\times(m^2-\mu^2)$. In the limit the cancellation produced by the unitary symmetry[23] is analogous to the conventional mass-renormalization in electrodynamics. The leading term of the remainder is analogous to the conventional wave-renormalization constant; here it would be finite even in perturbation theory (i.e., even without applying the Goldberger-Treiman procedure), since the fact that two different fields are involved appears throughout to have reduced the degree of divergence by one. Thus, a nonzero electromagnetic mixing of η and π^0 is predicted even in the exact unitary symmetry limit; by contrast it would appear intrinsically unreasonable to envisage situations in which either $m^2=\mu^2$ but the couplings are not given exactly by the eightfold way, or in which the couplings are given exactly by the eightfold way but $m^2\neq\mu^2$.

We give a provisional estimate[33] of the result of our dynamical calculation by using the eightfold way couplings[1]

$$\begin{aligned}
&g_{\pi NN}=g, && g_{\eta NN}=\sqrt{3}(1-\tfrac{4}{3}\alpha)g,\\
&g_{\pi\Sigma\Sigma}=2(1-\alpha)g, && g_{\eta\Sigma\Sigma}=(2\alpha/\sqrt{3})g,\\
&g_{\pi\Xi\Xi}=(1-2\alpha)g, && g_{\eta\Xi\Xi}=-\sqrt{3}(1-\tfrac{2}{3}\alpha)g,\\
&g_{\pi\Sigma\Lambda}=(2\alpha/\sqrt{3})g, && g_{\eta\Lambda\Lambda}=-(2\alpha/\sqrt{3})g,
\end{aligned} \tag{71}$$

to evaluate Eqs. (67) and (68); in the most reasonable range of values[34] $0.5\leqslant\alpha\leqslant0.9$ we find

$$R_\eta\sim(2\text{ to }4)\times10^{-3}, \tag{72}$$

$$R_\pi\sim-(1\text{ to }2)\times10^{-2}, \tag{73}$$

$$R\sim(-1.0\text{ to }-1.4)\times10^{-2}. \tag{74}$$

This result for R gives

$$\Gamma(\eta\to\pi^+\pi^-\pi^0)\sim140\text{ to }280\text{ eV} \tag{75}$$

for the width of the $\eta\to\pi^+\pi^-\pi^0$ decay when both the diagrams Fig. 1(a) and (b) are included. We note that

[32] S. P. Rosen, Phys. Rev. **132**, 1234 (1963), has pointed out that the contributions to the $\eta-\pi^0$ transition from various diagrams involving baryon-antibaryon pairs tend to cancel if the eightfold way symmetry is exact. His remarks do not apply to the mass difference contributions considered by us.

[33] We take the value $\delta M_\Xi\equiv M_{\Xi^-}-M_{\Xi^0}=+5$ MeV [J. Leitner, (private communication)]; we note that this is the value predicted by the eightfold way relation of Coleman and Glashow (Ref. 15). However, H. Schneider, Phys. Letters **4**, 360 (1963), found $M_{\Xi^0}=1325\pm5$ MeV, which gives $\delta M_\Xi=-5\pm5$ MeV.

[34] The couplings used in Ref. 3 are given by $g_D=\alpha g$, $g_F=(1-\alpha)g$.

this is not very different from the estimate (9) obtained by considering diagram (a) only, and using the strength of the $\eta-\pi^0$ transition predicted in terms of the electromagnetic mass splittings of the K and π. Comparison of our results (75) and (5) gives reasonable agreement with experiment[35] for the branching ratio $\Gamma(\eta\to 2\gamma)/\Gamma(\eta\to\pi^+\pi^-\pi^0)$.

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APPENDIX

We obtain a crude estimate of the ratio of the partial amplitude for the direct process $\eta\to N\bar{N}\to 3\pi$ to that for the boson pole processes considered in this paper. It is assumed that the order of magnitude of the four-boson couplings can reasonably, if roughly, be ascribed to latent baryon loops. Then the direct $\eta\to 3\pi$ coupling is estimated as $(\delta M/M)(\tilde{g}/g)4\pi\lambda(\eta\pi^0\pi\cdot\pi)$, where $\delta M/M$ enters as the cause of isospin violation, and $(\tilde{g}/g)4\pi\lambda$ by comparison with the ordinary $(\pi\cdot\pi)^2$ vertex, the latter also being pictured as induced by closed $N\bar{N}$ loops. Thus

$$|\mathfrak{M}(\eta\to\pi^+\pi^-\pi^0,\text{ direct})|\sim\frac{\delta M}{M}\frac{\tilde{g}}{g}8\pi\lambda. \quad \text{(A1)}$$

For the pole diagrams we have from Eqs. (19) and (74)

$$|\mathfrak{M}(\eta\to\pi^+\pi^-\pi^0,\text{ poles})|\sim|R(32\pi\lambda)|\sim 32\pi\lambda 10^{-2}. \quad \text{(A2)}$$

Hence,

$$\frac{|\mathfrak{M}(\text{direct})|}{|\mathfrak{M}(\text{poles})|}\sim\frac{\delta M/M}{10^{-2}}\frac{\tilde{g}}{g}\frac{1}{4}\sim 3\times 10^{-2}\frac{\tilde{g}}{g}. \quad \text{(A3)}$$

Current estimates of the eightfold way mixing parameter α indicate that $\tilde{g}/g$ is probably small; but even without this factor (A3) appears to make it plausible that the pole terms dominate. This reverses the assertion in the paper of Barton and Rosen[19] that nucleon loops do not naturally lead to pion pole dominance.

[35] C. Bacci, G. Penso, G. Salvini, A. Wattenberg, C. Mencuccini, R. Querzoli, and V. Silvestrini, Phys. Rev. Letters **11**, 37 (1963); F. S. Crawford, L. J. Lloyd, and E. C. Fowler, *ibid.* **10**, 546 (1963).